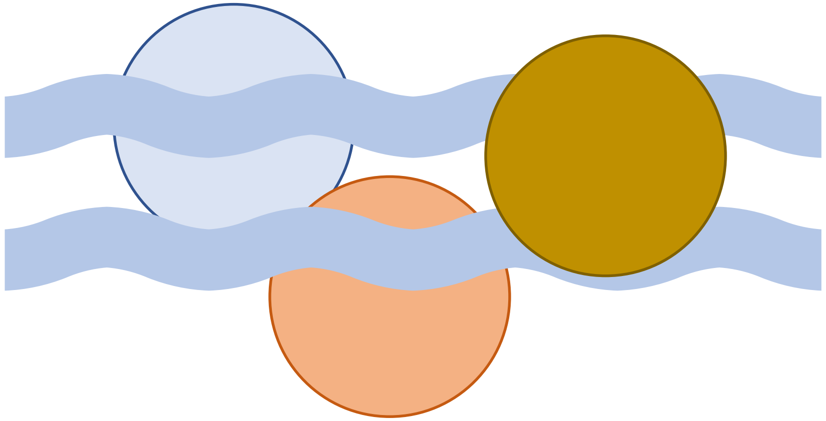
Mass Balance, Steady-State Soln, and Response Time



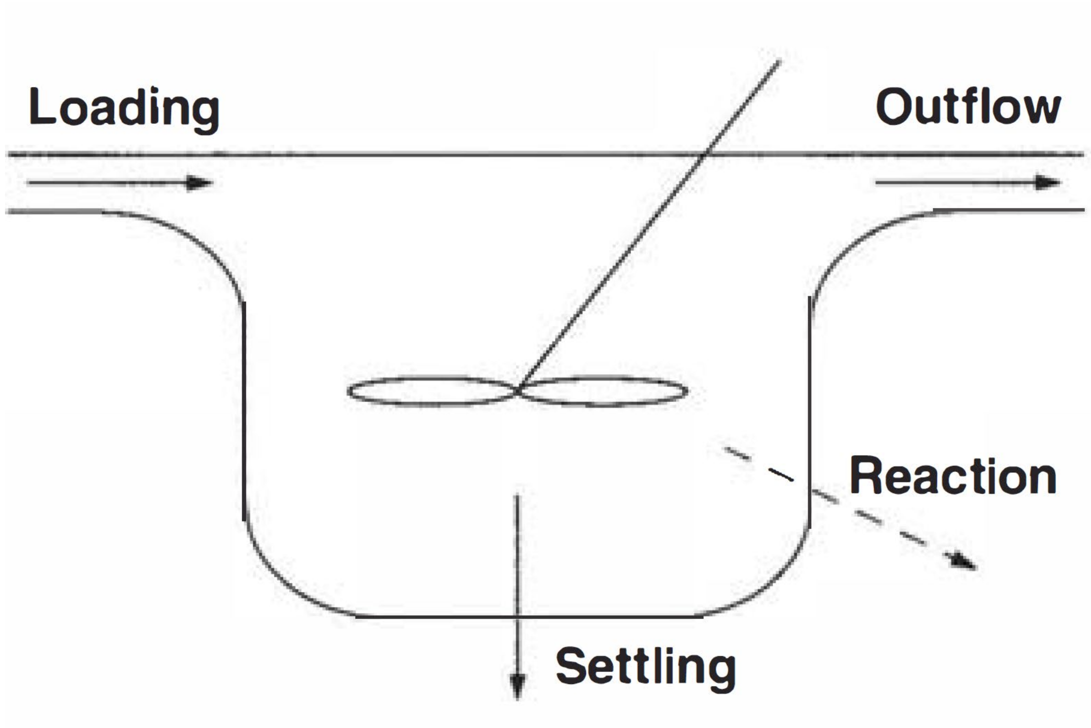
Mass Balance, Steady-State Soln, Response Time

Two most commonly posed questions in water quality modeling.

If we institute a treatment program:

-How much will the water body improve?

-How long will it take for the improvement to occur?



Mass Balance for Well-Mixed Lake

Steady-State Solutions

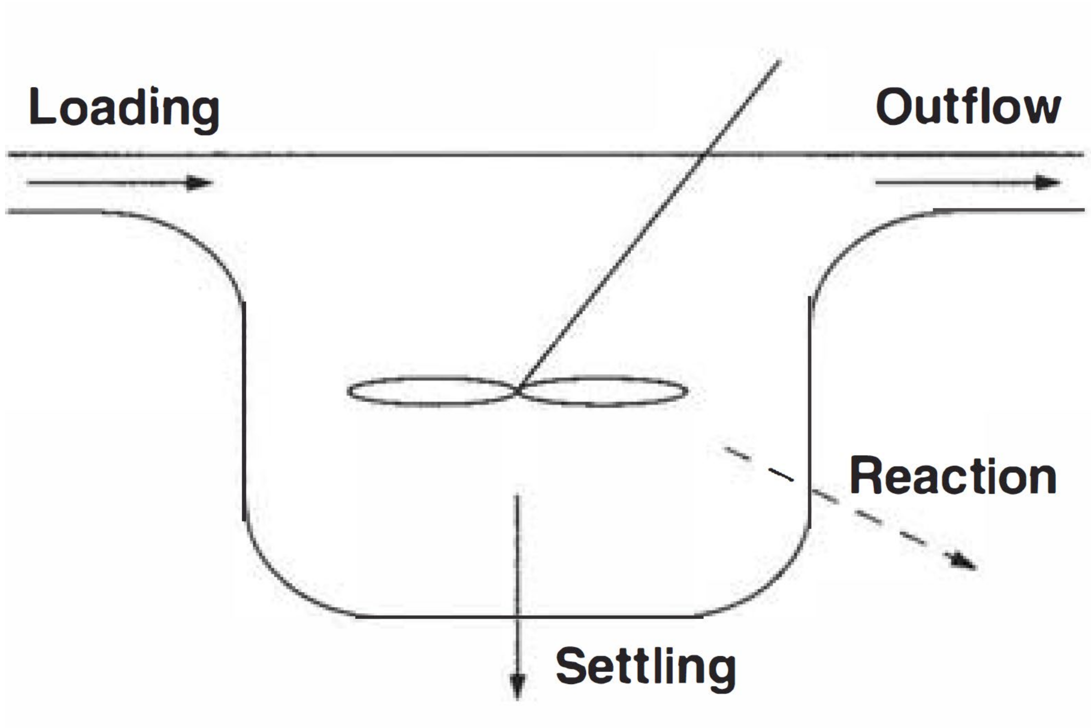
Transfer fns and Residence Time

Temporal Aspects of Pollutant Redxn

General Solution

Response Time

Mass Balance for a Well-Mixed Lake

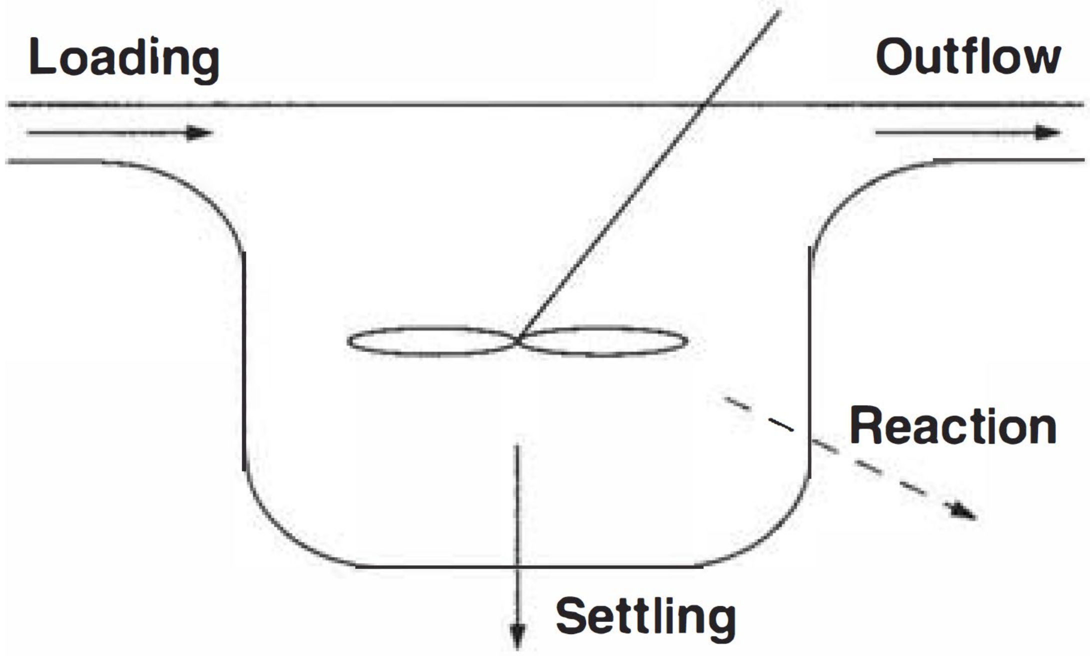


A completely mixed system, or a **continuously stirred tank reactor (CSTR)**, is among the simplest systems that can be used to model a natural water body.

A mass balance for a well-mixed lake. The arrows represent the major sources and sinks of the pollutant. The dashed arrow for the reaction sink is meant to distinguish it from the other sources and sinks, which are transport mechanisms.

$$Accumulation = loading - outflow - reaction - settling$$

Mass Balance for a Well-Mixed Lake



Accumulation: change in mass (M) over time

$$Accumulation = \frac{\Delta M}{\Delta t} = \frac{\Delta Vc}{\Delta t} = V\frac{dc}{dt}$$

Loading

$$Loading = W(t)$$

Outflow

$$Outflow = Qc$$

Reaction

$$Reaction = kM = kVc$$

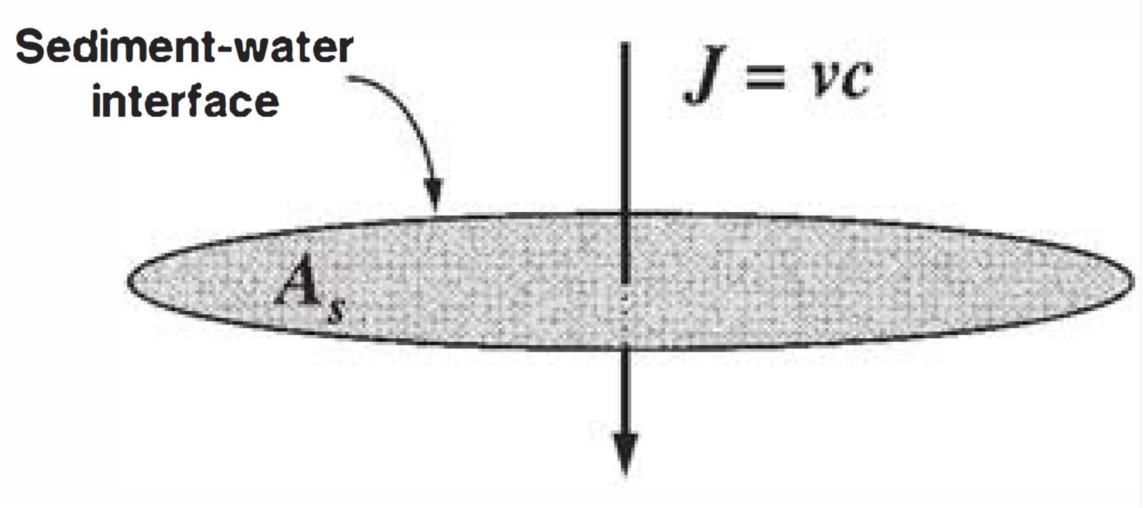
Settling

$$Settling = vA_s c$$

Total Balance

Terms can now be combined into the following mass balance for a well-mixed lake:

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_s c$$



Total Balance

Terms can now be combined into the following mass balance for a well-mixed lake:

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_sc$$

Dependent variable

Independent variable

Forcing function

Parameters:

V

Q

k

v

A_s

Steady-State Solutions

If the system is subject to a constant loading W for a sufficient time, it will attain a dynamic equilibrium condition called a ***steady-state***. The means that the accumulation is zero (i.e. $dc/dt = 0$):

$$c = \frac{W}{Q + kV + vA_s}$$

using $c = \frac{1}{a}W$

we have:

$$a = Q + kV + vA_s$$

This provides a formula that defines the assimilation factor in terms of measurable variables that reflect the system's physics, chemistry, and biology.

EXAMPLE 3.1. MASS BALANCE. A lake has the following characteristics:

$$\text{Volume} = 50{,}000 \text{ m}^3$$
$$\text{Mean depth} = 2 \text{ m}$$
$$\text{Inflow} = \text{outflow} = 7500 \text{ m}^3 \text{ d}^{-1}$$
$$\text{Temperature} = 25°\text{C}$$

The lake receives the input of a pollutant from three sources: a factory discharge of 50 kg d^{-1}, a flux from the atmosphere of 0.6 g m^{-2} d^{-1}, and the inflow stream that has a concentration of 10 mg L^{-1}. If the pollutant decays at the rate of 0.25 d^{-1} at 20°C ($\theta = 1.05$),

(*a*) Compute the assimilation factor.
(*b*) Determine the steady-state concentration.
(*c*) Calculate the mass per time for each term in the mass balance and display your results on a plot.

Steady-State Solutions

The major sources and sinks for a pollutant can be depicted as below:

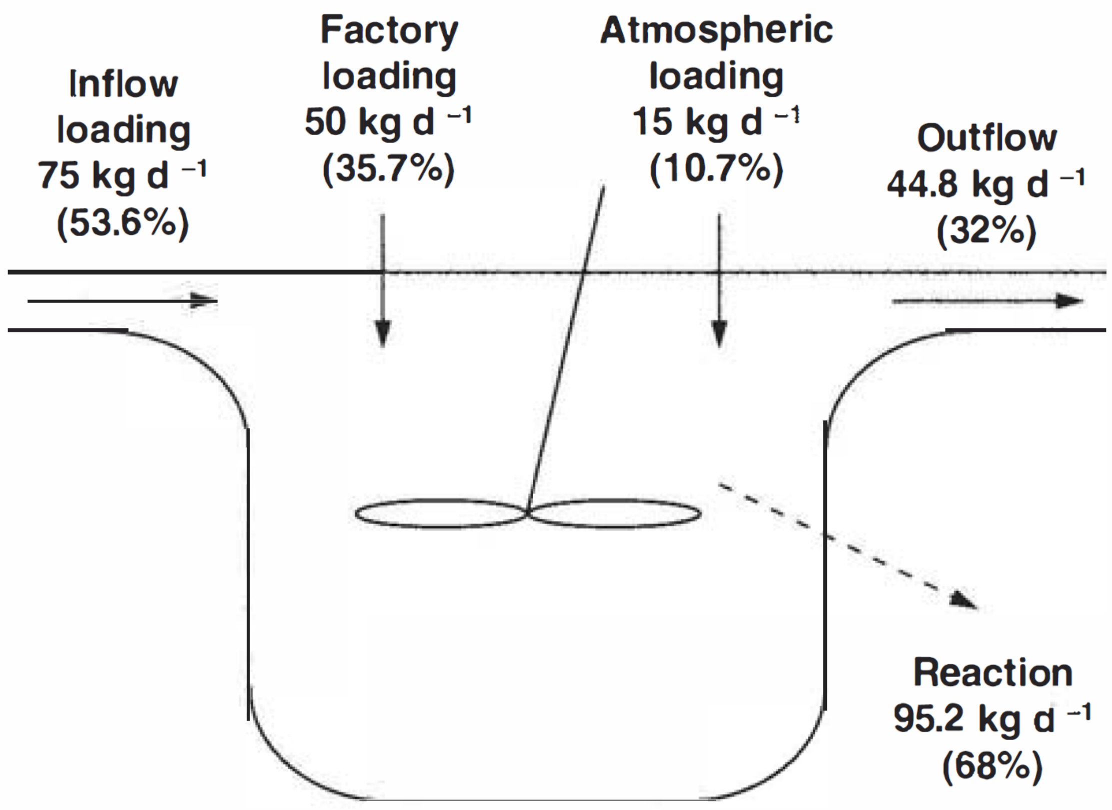


FIGURE 3.3
A mass balance for the well-mixed lake from Example 3.1. The arrows represent the major sources and sinks of the pollutant. The mass-transfer rates have also been included along with the percent of total mass inflow accounted for by each term.

Transfer Functions and Residence Time

Transfer function: An alternative way to formulate $c = \frac{W}{Q+kV+vA_s}$ is based on expressing the loading in the format of :

$$W = Qc_{in}$$

This equation can be substituted into the previous one and both numerator and denominator can be divided by c_{in} to yield:

$$\frac{c}{c_{in}} = \beta$$

Where β = the transfer function:

$$\beta = \frac{Q}{Q + kV + vA_s}$$

Transfer Functions and Residence Time

Residence time (τ_E) of a substance *E* represents the mean amount of time that a molecule or particle of *E* would stay or "reside" in a system.

It is defined for a steady-state, constant-volume system as (Stumm and Morgan 1981):

$$\tau_E = \frac{E}{|dE/dt|_{\pm}}$$

E.g. Residence time of water in a lake:

$$\tau_w = \frac{V}{Q}$$

Transfer Functions and Residence Time

We can use this relation to compute a “pollutant residence time”. For example the sinks can be represented on a mass basis:

$$\left|\frac{dM}{dt}\right|_{\pm} = Qc + kVc + vA_sc$$

and this can be utilized to yield:

$$\tau_c = \frac{V}{Qc + kVc + vA_sc}$$

EXAMPLE 3.2. TRANSFER FUNCTION AND RESIDENCE TIMES. For the lake in Example 3.1, determine the (*a*) inflow concentration, (*b*) transfer function, (*c*) water residence time, and (*d*) pollutant residence time.

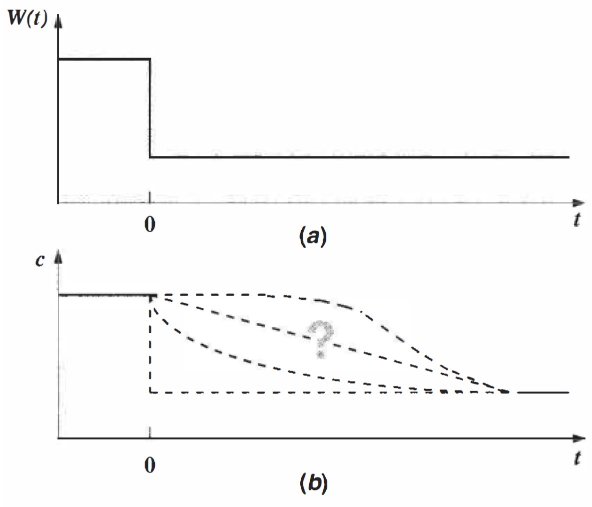
Temporal Aspects of Pollutant Reduction

After focusing on steady-state solutions, now we move to the temporal response of natural waters

Suppose for a steady-state system a waste removal project is implemented. We encounter two interrelated questions:

How long will it take for improved water quality to occur?

What will the “shape” of the recovery look like?



Temporal Aspects of Pollutant Reduction

After focusing on steady-state solutions, now we move to the temporal response of natural waters

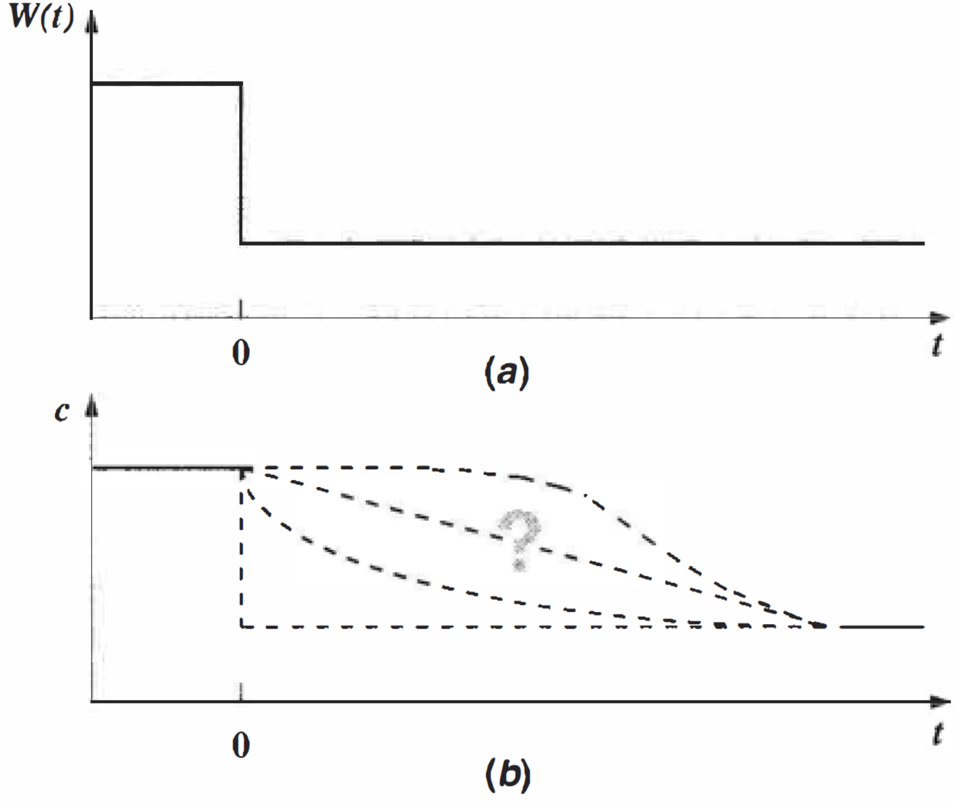
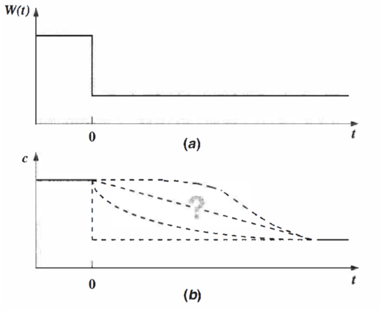


FIGURE 3.4
(*a*) A waste load reduction along with (*b*) four possible recovery scenarios for concentration.

Temporal Aspects of Pollutant Reduction

Let’s start with the mass-balance model:

$$V\frac{dc}{dt} = W(t) - Qc - kVc - vA_s c$$



Now we divide by volume to yield:

$$\frac{dc}{dt} = \frac{W(t)}{V} - \frac{Q}{V}c - kVc - \frac{v}{H}c$$

and collecting terms give:

$$\frac{dc}{dt} + \lambda c = \frac{W(t)}{V} \qquad where\ \lambda = \frac{Q}{V} + k + \frac{v}{H}$$

Temporal Aspects: The General Solution

If $c = c_0$ at $t = 0$, then with loading ($W(t)$) = 0, the steady-state equation ($\frac{dc}{dt} + \lambda c = \frac{W(t)}{V}$) can be solved by the separation of variables:

$$c = c_o e^{-\lambda t}$$

This provides an equation that describes how the lake's concentration changes as a function of time following the termination of waste loading ($W(t)$) .

Temporal Aspects: The General Solution

If c=co at t= 0, then with W(t) = 0 can be solved by the separation of variables:

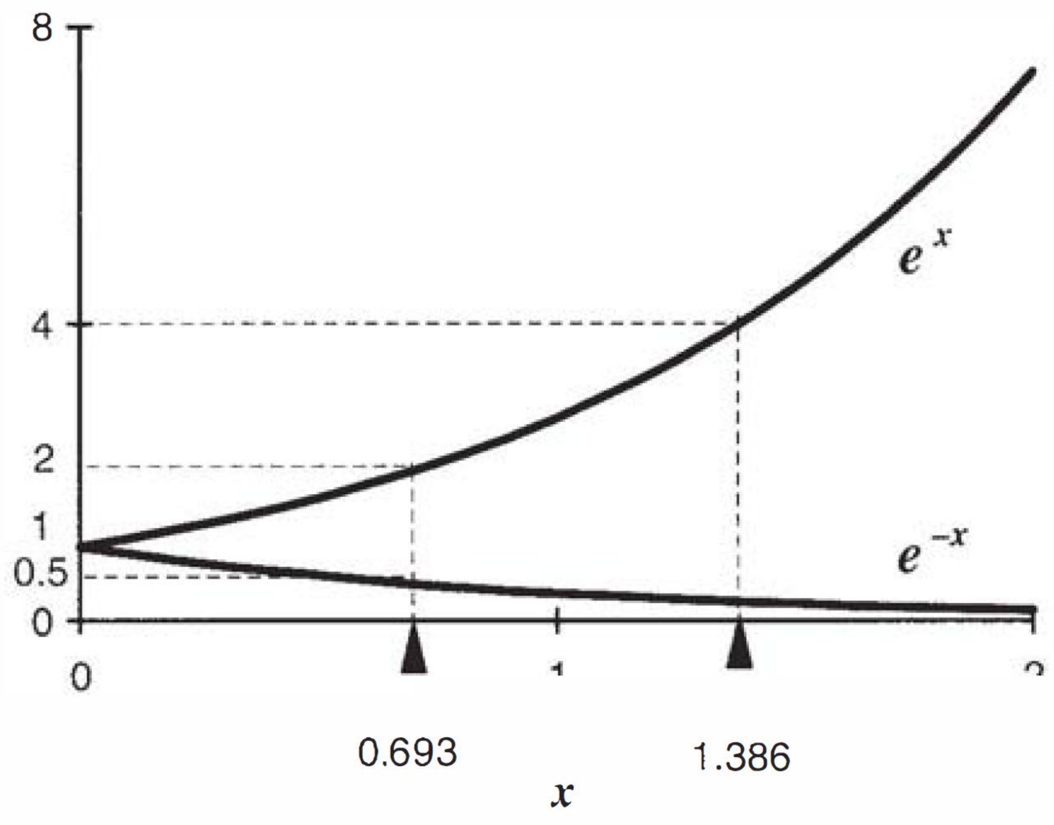


FIGURE 3.5
The exponential function.

Temporal Aspects: The General Solution

If c=co at t= 0, then with W(t) = 0 can be solved by the separation of variables:

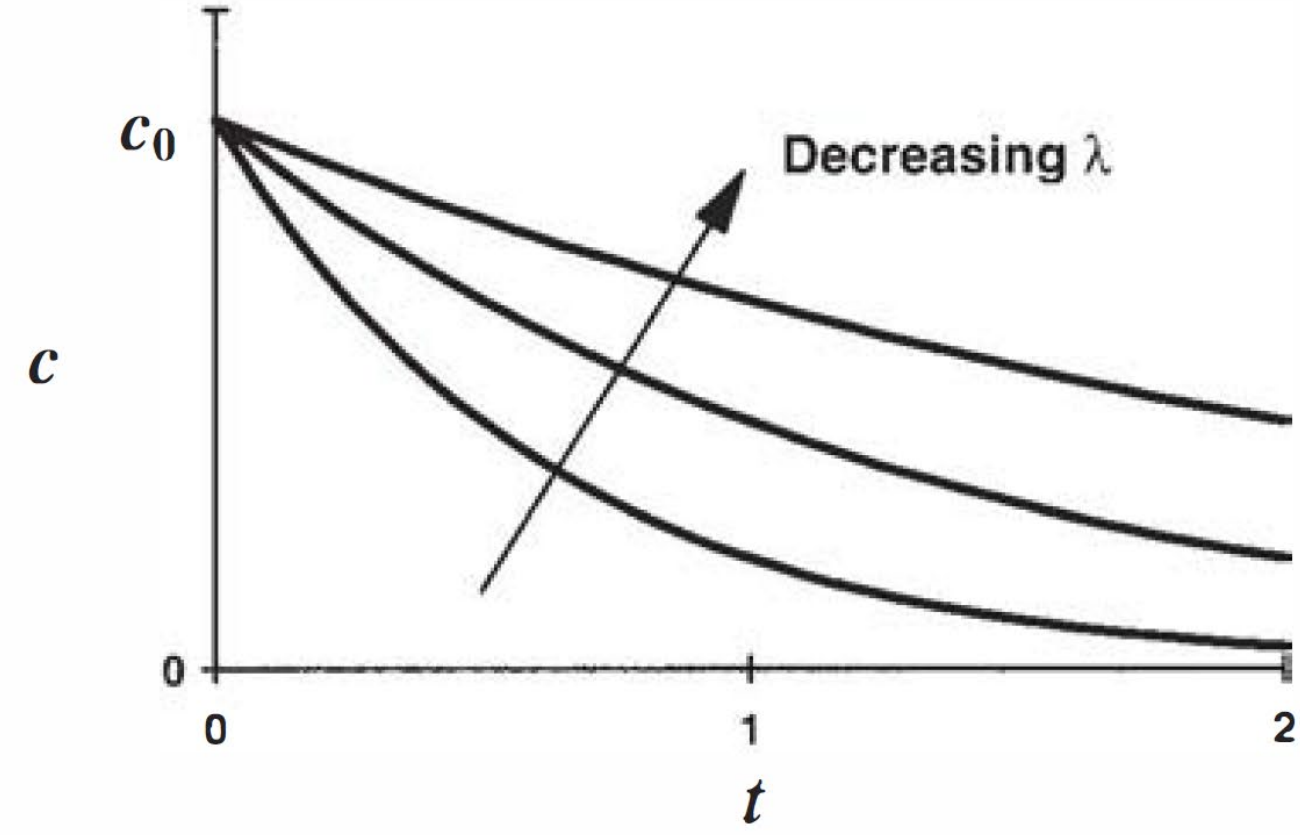


FIGURE 3.6
The temporal response of our well-mixed lake model following the termination of all loadings at $t = 0$.

EXAMPLE 3.3. GENERAL SOLUTION. In Example 3.1 we determined the steady-state concentration for a lake having the following characteristics:

Volume = 50,000 m^3	Temperature = 25°C
Mean depth = 2 m	Waste loading = 140,000 g d^{-1}
Inflow = outflow = 7500 m^3 d^{-1}	Decay rate = 0.319 d^{-1}

If the initial concentration is equal to the steady-state level (5.97 mg L^{-1}), determine the general solution.

Response Times

The response time parameter group represents the time it takes for the lake to complete a fixed percentage of its recovery.

Thus we can decide "how much" of the recovery is judged as being "enough" (e.g. 95% recovery could be satisfactory).

If we relate to [$c = c_o e^{-\lambda t}$] , a 50% response time means that the concentration is lowered to 50% of its initial value:

$$0.5c_o = c_o e^{-\lambda t_{50}}$$

Solving further:

$$e^{\lambda t_{50}} = 2 \qquad , or \qquad t_{50} = \frac{0.693}{\lambda}$$

Response Times

The response time parameter group represents the time it takes for the lake to complete a fixed percentage of its recovery.

$$t_\varphi = \frac{1}{\lambda} ln \frac{100}{100 - \varphi}$$

$$t_{95} = \frac{1}{\lambda} ln \frac{100}{100 - 95} = \frac{3}{\lambda}$$

TABLE 3.1
Response times

Response time	t_{50}	$t_{63.2}$	t_{75}	t_{90}	t_{95}	t_{99}
Formula	$0.693/\lambda$	$1/\lambda$	$1.39/\lambda$	$2.3/\lambda$	$3/\lambda$	$4.6/\lambda$

Response Times

The response time parameter group represents the time it tajes for the lake to complete a fixed percentage of its recovery.

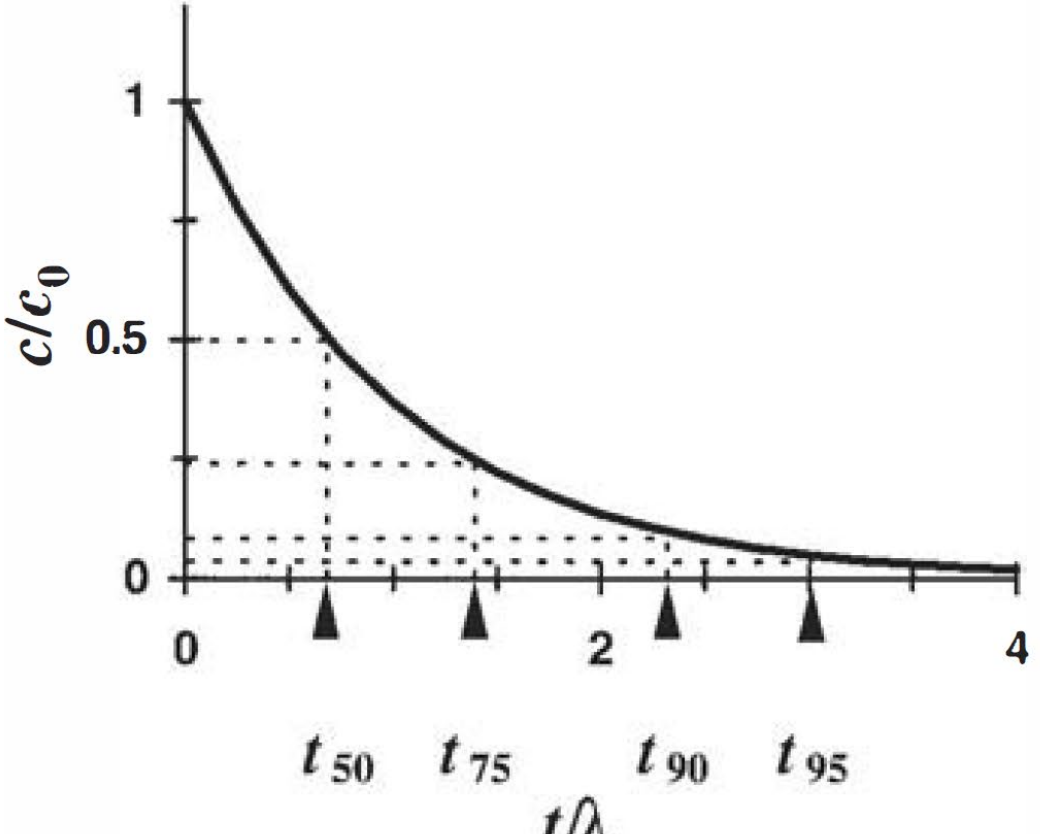


FIGURE 3.7
A plot of the general solution showing values of several response times.

EXAMPLE 3.4. RESPONSE TIME. Determine the 75%, 90%, 95%, and 99% response times for the lake in Example 3.3.